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A SIMPLE MODEL OF THE MECHANICAL BEHAVIOR OF CERAMIC-LIKE MATERIALS

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Abstract—This paper presents a macroscopic mechanical theory for ceramic-like materials undergoing isothermal deformations. The proposed model describes an elastic brittle material which is damageable only under tensile loading. The damage lowers the elastic stiffness in traction simulating hence the softening and the fracture (zero stiffness) of the material. The basic idea is to consider the continuum as a mixture of two phases—a linear elastic phase and a masonry phase (which shows a linear elastic behavior under compression but cannot hold tractive loads at all). The damage is then related to the volume fraction β of the elastic constituent. The constitutive relations are derived from macroscopic thermodynamics with the volume fraction β and its gradient $\nabla\beta$ taken as state variables.

1. INTRODUCTION

Ceramics are becoming very important as structural materials (Larsen, 1985; Devezas, 1984). They present some very interesting mechanical features such as high stiffness and predominantly elastic behavior. Nowadays, the field of applications for structural ceramics ranges from heat engine components to bearings and special scissors, for instance.

One of the limitations presented by these materials is the relatively low strength observed when they are subjected to tractive efforts (about one tenth of the strength under compressive loads).

In most ceramics we can verify a distribution of microcracks in an essentially elastic matrix. If the material were free of microcracks, it would behave as a linearly elastic material. On the other hand, a completely cracked elastic material becomes unable to equilibrate tensile loading but can continue to behave as an elastic material under compressive loadings (due to the mutual contact between the surfaces of the cracks).

In this paper, we study a mechanical model for elastic materials in which a distribution of microcracks is nucleated and grown, under high tractive solicitations, up to a macroscopic rupture. For the sake of simplicity we do not consider the degenerative behavior under compressive loadings. The main features of the proposed model are sketched in Fig. 1.

Many different macroscopic theories have been proposed to describe such kinds of brittle macrials. Among others, we can refer to the studies developed by Bazant and Pijaudier Cabot (1988), Bui *et al.* (1981), Florez (1989), Lemaitre and Chaboche (1985), Marigo (1981, 1985), Quoc Son (1984) and Simo and Ju (1987). In this paper we consider the continuum as a mixture of two phases—an elastic E-phase and a masonry M-phase. The macroscopic softening behavior is related to the volume fraction of the constituents. For compressive loadings the material has a linear elastic response since the two phases have the same behavior under compression. For the tractive solicitations, up to a limiting value of the elastic strain energy, the material behaves as a linear elastic one.

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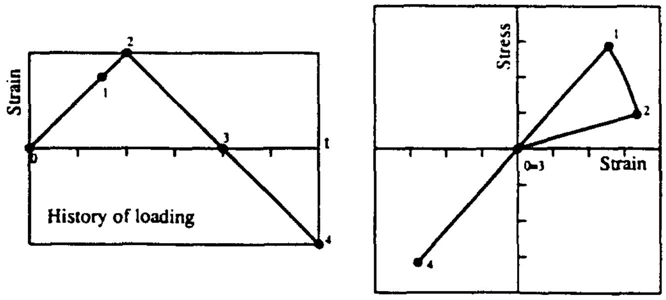

Fig. 1. Mechanical behavior described by the present model.

When this limit is reached, the volume fraction of the masonry constituent increases continuously up to the value one (and, consequently, the volume fraction of the elastic constituent decreases up to the value zero), when we assume that the material is locally "broken". Under this circumstance, the material cannot hold tractive loads at all. A study on the masonry-like materials can be found in Giaquinta and Giusti (1985).

For tractive solicitations, the mechanical behavior verified in this model is similar to the Continuum Damage Models presented in Bazant and Pijaudier Cabot (1988), Bui *et al.* (1981), Florez (1989), Fremond *et al.* (1990), Lemaitre and Chaboche (1985), Marigo (1981, 1985), Quoc Son (1984) and Simo and Ju (1987), and the volume fraction of the masonry phase can be related to the damage state variable adopted in these works. The difference resides in the fact that, in the present model the deterioration of the material due to tensile loading does not affect the stiffness under compressive loads. This behavior can be justified if we assume that, under compressive loads, the surfaces of the microcracks come into contact and can thus equilibrate the efforts from the neighboring material points. Further, a remarkable characteristic of the proposed model is given by the consideration of the gradient of the damage variable as a state variable, which is very important in order to have a mathematically correct description of the strain softening phenomenon.

The presentation of this paper can be outlined as follows: in Section 2, the dynamics of the medium are described by a generalization of the Principle of Virtual Power where both the strain ε and the volume fraction of the elastic phase are taken as "kinematical" variables. We consider that the material under consideration is a mixture of two other materials: an elastic one and a masonry one. Each of these materials is supposed to be a "phase" of the ceramic material. We define the volume fraction of the elastic phase as β, while the volume fraction of the masonry phase is denoted by $(1-\beta)$. In Section 3 we take into account the terms of the volumetric free energy Ψ which are responsible for the equilibrium between these two phases, as well as a volumetric pseudo potential of dissipation Φ which governs the evolution of β. The deterioration of the material under tractive solicitations is associated with the phase transformation from the elastic material to the masonry one. In Section 4, the model is shown to be in agreement with the Second Law of Thermodynamics.

2. THE CERAMIC MATERIAL AS A MIXTURE OF TWO PHASES

2.1. *The hypothesis of two phases: the kinematical variables and the Principle of Virtual Power*

In the present study we will model the ceramic material as a mixture of two phases: the *elastic* E-phase and the *masonry* M-phase. A model for masonry-like materials, which cannot equilibrate tensile loading but behave elastically under compressible loadings, was studied by Giaquinta and Giusti (1985), in the context of plane stresses. The behavior of both phases is illustrated in Fig. 2.

For each material point, we can define a volume fraction $\beta_E = \beta$ of the E-phase and a volume function $\beta_M = (1-\beta)$ of the M-phase. The process of nucleation and growth of microcracks can be macroscopically characterized by the phase transformation from the

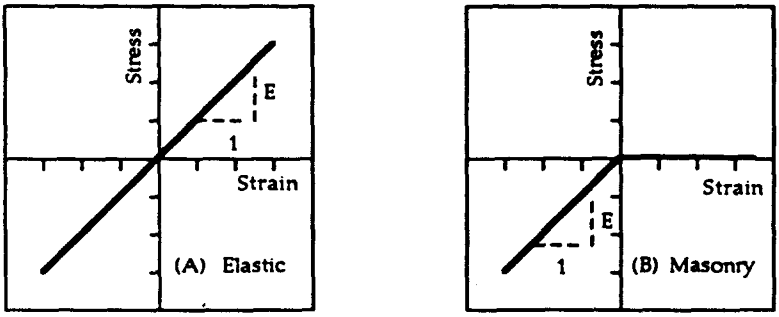

Fig. 2. Mechanical behavior of elastic and masonry materials.

E-phase to the M-phase. For each phase, the strain is a state variable. We shall assume that at each material point, the strains ε_E and ε_M are the same, which we shall denote ε. Furthermore, the proportion of each phase in the material, characterized by the parameter β, is also considered to be a state variable.

If both state variables ε and β are supposed to be kinematical variables [following the ideas presented by Frémond (1987) and Truong Dinh Tien (1990) in the context of adhesion], then a First Gradient approach for the Principle of Virtual Power gives:

$$-\int_{\Omega}\sigma\cdot\nabla\mathbf{v}+\int_{\partial\Omega_1}\mathbf{f}\cdot\mathbf{v}+\int_{\Omega}\mathbf{b}\cdot\mathbf{v}=\int_{\Omega}\rho\ddot{\mathbf{u}}\cdot\mathbf{v}\quad\forall\mathbf{v}\in V, \tag{1}$$

$$\int_{\Omega}(F\gamma+\mathbf{H}\cdot\nabla\gamma)-\int_{\partial\Omega}g\gamma-\int_{\Omega}l\gamma=0\quad\forall\gamma\in\Gamma, \tag{2}$$

where Ω is the region occupied by the body, $\mathbf{v}:\Omega\to R^3$ is an element of the set V of the virtual variations on the displacement fields $\mathbf{u}:\Omega\to R^3$, $\mathbf{f}:\partial\Omega_1\to R^3$ represents the contact forces on the boundary $\partial\Omega_1$ of Ω, $\mathbf{b}:\Omega\to R^3$ is a distance force, and $\gamma:\Omega\to R$ is an element of the set Γ of the virtual variations on the phase parameter $\beta:\Omega\to R$. The fields $F:\Omega\to R$ and $\mathbf{H}:\Omega\to \mathrm{R}^3$ are the zeroth and the first order internal thermodynamical forces respectively associated with the fields $\beta:\Omega\to R$ and $\nabla\beta:\Omega\to R^3$, while $g:\partial\Omega\to R$ and $l:\Omega\to R$ are the external thermodynamical forces associated with β respectively applied on the surface $\partial\Omega$ and on the domain Ω. In the present work we do not consider the inertial effects associated with the parameter β.

From (1) and (2) we can obtain the following local equilibrium equations and boundary conditions:

$$\operatorname{div}\sigma+b=\rho\ddot{u},\quad \sigma=\sigma^{\mathrm{T}}\quad\forall\,\mathrm{x}\in\Omega, \tag{3}$$

$$\sigma\mathbf{n}=\mathbf{f}\quad\forall\mathrm{x}\in\partial\Omega_1, \tag{4}$$

$$-F+\operatorname{div}\mathbf{H}+l=0\quad\forall\mathrm{x}\in\Omega, \tag{5}$$

$$g=H\cdot n\quad\forall\mathrm{x}\in\partial\Omega, \tag{6}$$

where $\mathbf{n}(\mathrm{x})$, $\mathrm{x}\in\partial\Omega$, is the unitary outward normal to the boundary $\partial\Omega$, at x.

In the above expressions (1)–(6), the thermodynamical forces $\sigma:\Omega\to\mathrm{Sym}^3$, $F:\Omega\to R$ and $\mathrm{H}:\Omega\to R^3$ are defined for the mixture. Here, Sym^3 denotes the set of symmetric linear operators from R^3 to R^3.

2.2. *The First and Second Laws of Thermodynamics*

The First Law of Thermodynamics for this kind of continuum can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega}(e+\tfrac{1}{2}\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}) = \int_{\partial\omega}[(\boldsymbol{\sigma}\mathbf{n})\cdot\dot{\mathbf{u}}+(\mathbf{H}\cdot\mathbf{n})\dot{\beta}-\mathbf{q}\cdot\mathbf{n}]+\int_{\omega}(\mathbf{b}\cdot\dot{\mathbf{u}}+l\cdot\dot{\beta}) \quad \forall\omega\subset\Omega, \tag{7}$$

where $e:\Omega\rightarrow R$ is the volumetric internal energy, $\rho:\Omega\rightarrow R$ is the density of the material and $\mathbf{q}:\Omega\rightarrow R^3$ is the heat flux. We must warn of the fact that the kinetic energy associated with β was not considered here.

The associated local form of the First Law is given by:

$$\dot{e} = \boldsymbol{\sigma}\cdot\nabla\dot{\mathbf{u}}+F\dot{\beta}+\mathbf{H}\cdot\nabla\dot{\beta}-\operatorname{div}\mathbf{q} \quad \forall\mathbf{x}\in\Omega. \tag{8}$$

Similar forms of the First Law also arise in the microstructure theories of Mindlin (1964), Toupin (1964) or Goodman and Cowin (1972).

The Second Law of Thermodynamics is expressed by the inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega}s \geqslant -\int_{\partial\omega}\frac{\mathbf{q}\cdot\mathbf{n}}{\theta} \quad \forall\omega\subset\Omega, \tag{9}$$

where $s:\Omega\rightarrow R$ is the volumetric entropy and $\theta:\Omega\rightarrow R$ is the absolute temperature. The local version of the Second Law is given by:

$$\theta\dot{s}+\operatorname{div}\mathbf{q}-\frac{\mathbf{q}}{\theta}\cdot\nabla\theta \geqslant 0 \quad \forall x\in\Omega. \tag{10}$$

The substitution of div q, computed from (8), in the above inequality leads to:

$$\boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}+F\dot{\beta}+\mathbf{H}\cdot\nabla\dot{\beta}+\theta\dot{s}-\dot{e}-\frac{\mathbf{q}}{\theta}\cdot\nabla\theta \geqslant 0 \quad \forall x\in\Omega, \tag{11}$$

where the equality $\boldsymbol{\sigma}\cdot\nabla\dot{\mathbf{u}} = \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}$ comes from the symmetry of $\boldsymbol{\sigma}$.

If we write the inequality (11) in terms of the Helmholtz specific free energy $\Psi = e-\theta s$, we obtain the Clausius–Duhem inequality given by:

$$d = \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}+F\dot{\beta}+\mathbf{H}\cdot\nabla\dot{\beta}-\dot{\Psi}-s\dot{\theta}-\frac{\mathbf{q}}{\theta}\cdot\nabla\theta \geqslant 0 \quad \forall\mathbf{x}\in\Omega. \tag{12}$$

In the present study we shall consider only isothermal processes so that the above version of the Second Law reduces to:

$$d = \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}+F\dot{\beta}+\mathbf{H}\cdot\nabla\dot{\beta}-\dot{\Psi} \geqslant 0 \quad \forall\mathbf{x}\in\Omega. \tag{13}$$

The local version (12), (13) of the Second Law of Thermodynamics defines the rate of dissipation d and makes a distinction between possible processes ($d \geqslant 0$) and impossible processes ($d < 0$).

2.3. *The constitutive equation for the stress σ*

Based on the assumption that the material under consideration is a mixture of two phases, its behavior is supposed to comprise a combination of the mechanical properties of both constituents. Let us first study the constitutive laws for each phase present in the material.

2.3.1. *The E-phase*. The E-phase is simply an isotropic elastic material. Under the hypothesis of isothermal and small deformations, the local state of an elastic material is supposed to be only a function of the strain ε. The free energy potential Ψ_E is supposed to have the following form:

$$\Psi_E(\varepsilon) = \frac{\lambda}{2}(\mathrm{tr}\,\varepsilon)^2 + \mu\,\mathrm{tr}(\varepsilon^2), \tag{14}$$

where λ and μ are the Lamé constants, or by the complementary free energy potential:

$$\Psi_E^C(\sigma_E) = \sup_{\varepsilon}\{\sigma_E\cdot\varepsilon - \Psi_E\} = \frac{\lambda^C}{2}(\mathrm{tr}\,\sigma_E)^2 + \mu^C\,\mathrm{tr}(\sigma_E^2), \tag{15}$$

where

$$\lambda^C = -\frac{\lambda}{2\mu(3\lambda+2\mu)} \quad \text{and} \quad \mu^C = \frac{1}{4\mu}. \tag{16}$$

From (14) and (15), the following equivalent constitutive relations are postulated:

$$\sigma_E = \frac{\partial\Psi_E}{\partial\varepsilon} = \lambda(\mathrm{tr}\,\varepsilon)1 + 2\mu\varepsilon, \tag{17}$$

$$\varepsilon = \frac{\partial\Psi_E^C}{\partial\sigma_E} = \lambda^C(\mathrm{tr}\,\sigma_E)1 + 2\mu^C\sigma_E. \tag{18}$$

2.3.2. *The M-phase*. The M-phase is basically an elastic material which cannot achieve positive values for the stress σ_M. As we are considering a constraint on σ_M, it is easier to design the constitutive behavior for this phase from the complementary potential approach, when we have simply:

$$\Psi_M^C = \frac{\lambda^C}{2}(\mathrm{tr}\,\sigma_M)^2 + \mu^C\,\mathrm{tr}(\sigma_M^2) + I\ (\sigma_M), \tag{19}$$

where $I\ (\sigma)$ is the indicator function of the convex set of stress tensors with nonpositive eigenvalues σ_{M_i}, $i = 1, 2, 3$, of the stress tensor σ_M such that:

$$I\ = \begin{cases} 0 & \text{if } \sigma_M \in \Sigma, \\ +\infty & \text{otherwise}, \end{cases} \tag{20}$$

where Σ is the following convex set:

$$\Sigma = \{\sigma_M \in \mathrm{Sym}^3;\ \sigma_{M_i} \leqslant 0,\ i = 1, 2, 3\}. \tag{21}$$

From (19) we obtain the constitutive relation expressed by:

$$\varepsilon(\sigma_M) \in \partial\Psi_M^C(\sigma_M) = \lambda^C(\mathrm{tr}\,\sigma_M)1 + 2\mu^C\sigma_M + \partial I\ (\sigma_M), \tag{22}$$

where $\partial I\ (\sigma_M)$ is the subdifferential [see e.g. Ekeland and Teman (1976)] of the function $I\ (\sigma_M)$, at σ_M, given by:

$$\partial I\ (\sigma_M) = \{\mathbf{h} \in \mathrm{Sym}^3;\ \mathbf{h}\cdot(\tau - \sigma_M) \leqslant I\ (\tau) - I\ (\sigma_M) \quad \forall \tau \in \mathrm{Sym}^3\}. \tag{23}$$

The term $I\ (\sigma)$ is associated with the idea that a stress state with positive eigenvalues would imply an infinite value for the complementary energy Ψ_M^C, which cannot be attained.

Let us analyse eqn (22) as a unidimensional model: if σ_M is negative, the set $\partial I_-(\sigma_M)$ reduces to zero and the constitutive relation is the same as for the linear elasticity. For $\sigma_M = 0$, the strain ε can achieve any non-negative value, since the set $\partial I_-(0)$ contains all these elements. The positive strain associated with a null stress represents the measure of the gap between the surfaces of the crack opened due to the positive strain. Equation (22) is not defined for positive values of σ_M, since then the set $\partial I_-(\sigma_M)$ is void, so that the material cannot attain positive stresses. For the three-dimensional case, some care must be taken in the corresponding analysis, since some positive values of the strain are not associated with the separation between the surfaces of the cracks, but rather they are due to the Poisson's effect.

Since the functional Ψ_M^C is convex, we can assume that it is the dual of a convex lower semi-continuous free energy potential Ψ_M and hence:

$$
\begin{aligned}
\Psi_M(\varepsilon) &= \sup_{\sigma_M}\{\sigma_M\cdot\varepsilon-\Psi_M^C\}\\
&= \sup_{\sigma_M}\left\{\sigma_M\cdot\varepsilon-\frac{\lambda^C}{2}(\mathrm{tr}\,\sigma_M)^2-\mu^C\,\mathrm{tr}\,(\sigma_M^2)-I_-(\sigma_M)\right\}\\
&= \sup_{\sigma_M\in\Sigma}\left\{\sigma_M\cdot\varepsilon-\frac{\lambda^C}{2}(\mathrm{tr}\,\sigma_M)^2-\mu^C\,\mathrm{tr}\,(\sigma_M^2)\right\}. \qquad (24)
\end{aligned}
$$

From the development presented in the Appendix, the masonry material is shown to be isotropic and therefore the Transfer Theorem (A4) enables the representation of (24) in terms of the eigenvalues ε_i of ε and σ_{M_i} of σ_M since their eigenvectors are collinear:

$$
\begin{aligned}
\Psi_M(\varepsilon) &= \sup_{\sigma_M\in\Sigma}\left\{\sigma_M\cdot\varepsilon-\frac{\lambda^C}{2}(\mathrm{tr}\,\sigma_M)^2-\mu^C\,\mathrm{tr}\,(\sigma_M^2)\right\}\\
&= \sup_{\sigma_M\in\Sigma} w_\varepsilon(\sigma_M), \qquad (25)
\end{aligned}
$$

where

$$
w_\varepsilon(\sigma_M) = \sigma_{M_1}\varepsilon_1+\sigma_{M_2}\varepsilon_2+\sigma_{M_3}\varepsilon_3-\frac{\lambda^C}{2}(\sigma_{M_1}+\sigma_{M_2}+\sigma_{M_3})^2-\mu^C(\sigma_{M_1}^2+\sigma_{M_2}^2+\sigma_{M_3}^2). \qquad (26)
$$

The computation of (25) presents distinct features depending on the strain ε considered. Let us analyse each one of these situations:

(a) Let us suppose initially that the strain ε is such that the supremum of (25) can be computed without the activation of any of the constraints from Σ. In this instance, the required supremum is also the stationary point $\sigma_M^{S_0}$ of the functional w_ε, obtained from the condition:

$$
\frac{\partial}{\partial\sigma_{M_i}} w_\varepsilon(\sigma_M^{S_0}) = 0, \quad i = 1,2,3, \qquad (27)
$$

which gives:

$$
\varepsilon_i = \lambda^C(\sigma_{M_1}^{S_0}+\sigma_{M_2}^{S_0}+\sigma_{M_3}^{S_0})+2\mu^C\sigma_{M_i}^{S_0}, \quad i = 1,2,3, \qquad (28)
$$

or the inverse relation:

$$
\sigma_{M_i}^{S_0}(\varepsilon) = \lambda(\varepsilon_1+\varepsilon_2+\varepsilon_3)+2\mu\varepsilon_i, \quad i = 1,2,3. \qquad (29)
$$

The substitution of (29) into (26) gives directly the free energy $\Psi_M(\varepsilon)$:

$$\Psi_{\mathrm{M}}(\boldsymbol{\varepsilon}) = \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{\varepsilon})^2 + \mu \operatorname{tr}(\boldsymbol{\varepsilon}^2) \tag{30}$$

provided that the strain is such that $\sigma_{\mathrm{M}_i}^{S_0} \leqslant 0$, $i = 1, 2, 3$ or, from (29), that:

$$\varepsilon_i \leqslant -\frac{\lambda}{\lambda+2\mu}(\varepsilon_j+\varepsilon_k), \quad (i,j,k) \in P, \tag{31}$$

where $P = \{(1,2,3), (2,3,1), (3,1,2)\}$ is the set of cyclic permutations of the subscripts 1, 2 and 3. The inequalities (31) define the set:

$$E_0 = \left\{\boldsymbol{\varepsilon} \in \mathrm{Sym}^3 ; \varepsilon_i \leqslant -\frac{\lambda}{\lambda+2\mu}(\varepsilon_j+\varepsilon_k), \quad (i,j,k) \in P\right\}, \tag{32}$$

where (30) holds.

From (30), we obtain the following stress–strain relations:

$$\boldsymbol{\sigma} = \frac{\partial \Psi_{\mathrm{M}}}{\partial \boldsymbol{\varepsilon}} = \lambda(\operatorname{tr}\boldsymbol{\varepsilon})\underline{1} + 2\mu\boldsymbol{\varepsilon} \quad \forall \boldsymbol{\varepsilon} \in E_0. \tag{33}$$

Since the strains belonging to E_0 must verify the inequalities (31), we conclude from (33) that the associated stresses $\boldsymbol{\sigma}_{\mathrm{M}}$ are such that their eigenvalues are all non-positive.

(b) Now we consider the situations where one of the constraints, say $\sigma_{\mathrm{M}_k} \leqslant 0$, is activated during the search for the supremum of w_ε. In this case, the supremum is located on the boundary $\sigma_{M_k} = 0$, where we can write w_ε simply as:

$$w_\varepsilon(\boldsymbol{\sigma}_{\mathrm{M}})_{\sigma_{\mathrm{M}_k}=0} = \sigma_{\mathrm{M}_i}\varepsilon_i + \sigma_{\mathrm{M}_j}\varepsilon_j - \frac{\lambda^{\mathrm{C}}}{2}(\sigma_{\mathrm{M}_i}+\sigma_{\mathrm{M}_j})^2 - \mu^{\mathrm{C}}(\sigma_{\mathrm{M}_i}^2+\sigma_{\mathrm{M}_j}^2). \tag{34}$$

The stationary point $\boldsymbol{\sigma}_{\mathrm{M}}^{S_1}$ of (34) is given by:

$$\sigma_{\mathrm{M}_q}^{S_1} = \lambda_1(\varepsilon_i+\varepsilon_j) + 2\mu_1\varepsilon_q, \quad q = i, j, \tag{35}$$

where

$$\lambda_1 = -\frac{\lambda_{\mathrm{C}}}{4\mu^{\mathrm{C}}(\lambda^{\mathrm{C}}+\mu^{\mathrm{C}})} = \frac{2\lambda\mu}{\lambda+2\mu}, \quad \mu_1 = \frac{1}{4\mu^{\mathrm{C}}} = \mu \tag{36}$$

and therefore the supremum of (34) is given by:

$$\Psi_{\mathrm{M}}(\boldsymbol{\varepsilon}) = \frac{\lambda_1}{2}(\varepsilon_i+\varepsilon_j)^2 + \mu_1(\varepsilon_i^2+\varepsilon_j^2) \tag{37}$$

for any strain satisfying the inequalities $\sigma_{\mathrm{M}_k}^{S_0}(\boldsymbol{\varepsilon}) > 0$ and $\sigma_{\mathrm{M}_q}^{S_1}(\boldsymbol{\varepsilon}) \leqslant 0$, $q = i, j$, $(i,j,k) \in P$. These constraints define the set:

$$E_1 = \left\{\boldsymbol{\varepsilon} \in \mathrm{Sym}^3 ; \varepsilon_k > -\frac{\lambda}{\lambda+2\mu}(\varepsilon_i+\varepsilon_j), \quad \varepsilon_i \leqslant -\frac{\lambda_1}{\lambda_1+2\mu_1}\varepsilon_j, \right.$$
$$\left. \varepsilon_j \leqslant -\frac{\lambda_1}{\lambda_1+2\mu_1}\varepsilon_i, \quad (i,j,k) \in P\right\}, \tag{38}$$

where the relations (29) and (35) were taken into account.

Since the free energy (37) is an isotropic function of $\boldsymbol{\varepsilon}$, we can write the stress–strain relations as:

$$\sigma_{M_q} = \frac{\partial \Psi_M(\boldsymbol{\varepsilon})}{\partial \varepsilon_q} = \lambda_1(\varepsilon_i + \varepsilon_j) + 2\mu_1 \varepsilon_q \leqslant 0, \quad q = i, j, \qquad (i, j, k) \in P.$$
$$\sigma_{M_k} = \frac{\partial \Psi_M(\boldsymbol{\varepsilon})}{\partial \varepsilon_k} = 0, \tag{39}$$

(c) Let the strain $\boldsymbol{\varepsilon}$ be such that two of the constraints, say $\sigma_{M_j} \leqslant 0$ and $\sigma_{M_k} \leqslant 0$, are active in the computation of (25). In this instance, the supremum of w_ε belongs to the manifold characterized by the equalities $\sigma_{M_j} = 0$ and $\sigma_{M_k} = 0$, where w_ε behaves as:

$$w_\varepsilon(\sigma_M)_{\sigma_{M_j} = \sigma_{M_k} = 0} = \sigma_{M_i}\varepsilon_i - \tfrac{1}{2}(\lambda^C + 2\mu^C)\sigma_{M_i}^2. \tag{40}$$

The stationary point of the above expression is characterized by:

$$\sigma_{M_i}^{S_2} = \frac{1}{\lambda^C + 2\mu^C}\,\varepsilon_i = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\,\varepsilon_i \tag{41}$$

and thus the substitution of (41) in (40) gives the free energy functional:

$$\Psi_M(\boldsymbol{\varepsilon}) = \frac{1}{2}\,\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\,\varepsilon_i^2 \tag{42}$$

for any strain belonging to the set:

$$E_2 = \left\{ \boldsymbol{\varepsilon} \in \mathrm{Sym}^3 \,;\, \varepsilon_i \leqslant 0, \quad \varepsilon_j > -\frac{\lambda}{2(\lambda + \mu)}\varepsilon_i, \quad \varepsilon_k > -\frac{\lambda}{2(\lambda + \mu)}\varepsilon_i, \quad (i, j, k) \in P \right\}. \tag{43}$$

(d) Finally, we have to consider the situation where all the constraints of Σ are active in (25). In this case, the supremum is located at the point $\sigma_M = \mathbf{0}$ and we have simply:

$$\Psi_M(\boldsymbol{\varepsilon}) = 0, \quad \forall \boldsymbol{\varepsilon} \in E_3, \tag{44}$$

where

$$E_3 = \{ \boldsymbol{\varepsilon} \in \mathrm{Sym}^3 \,;\, \varepsilon_i > 0, i = 1, 2, 3 \}. \tag{45}$$

The corresponding stress is clearly null.

In conclusion, we have the free energy potential $\Psi_M(\boldsymbol{\varepsilon})$ given by:

$$\Psi_M(\boldsymbol{\varepsilon}) = \begin{cases} \dfrac{\lambda}{2}(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 + \mu\,\mathrm{tr}\,(\boldsymbol{\varepsilon}^2), & \text{if } \boldsymbol{\varepsilon} \in E_0, \\[2ex] \dfrac{\lambda\mu}{\lambda + 2\mu}(\varepsilon_i + \varepsilon_j)^2 + \mu(\varepsilon_i^2 + \varepsilon_j^2), & \text{if } \boldsymbol{\varepsilon} \in E_1, \\[2ex] \dfrac{1}{2}\,\dfrac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\,\varepsilon_i^2, & \text{if } \boldsymbol{\varepsilon} \in E_2, \\[2ex] 0, & \text{if } \boldsymbol{\varepsilon} \in E_3, \end{cases} \tag{46}$$

and the stress–strain relations can be written as:

$$
\begin{aligned}
&\sigma_M(\varepsilon) = \lambda \operatorname{tr} \varepsilon \underline{1} + 2\mu\varepsilon && \forall \varepsilon \in E_0, \\
&\sigma_{M_q}(\varepsilon) = \frac{2\lambda\mu}{\lambda+2\mu}(\varepsilon_i+\varepsilon_j)+2\mu\varepsilon_q, \quad q = i,j, \quad \sigma_{M_k}(\varepsilon) = 0 && \forall \varepsilon \in E_1, \\
&\sigma_{M_i}(\varepsilon) = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\varepsilon_i, \quad \sigma_{M_j}(\varepsilon) = 0, \quad \sigma_{M_k}(\varepsilon) = 0 && \forall \varepsilon \in E_2, \\
&\sigma_M(\varepsilon) = 0 && \forall \varepsilon \in E_3.
\end{aligned} \tag{47}
$$

2.3.3. *The two-phase material.* Based on the assumption that the ceramic material is a mixture of the E-phase and the M-phase, we define the following free energy potential for this new material:

$$
\Psi(\varepsilon, \beta, \nabla\beta) = \beta\Psi_E(\varepsilon) + (1-\beta)\Psi_M(\varepsilon) + \chi(\beta, \nabla\beta), \tag{48}
$$

where β describes the volume fraction of the E-phase and $\chi(\beta, \nabla\beta)$ is a functional associated to the state law for the parameters β and $\nabla\beta$, as will be made clear in the next paragraph. We recall that in (48) the strain variables of the two materials are supposed to assume the same values in both phases, at each material point.

From (48), we can write the stress–strain relation for the mixture as:

$$
\sigma = \frac{\partial\Psi(\varepsilon, \beta, \nabla\beta)}{\partial\varepsilon} = \beta\sigma_E(\varepsilon) + (1-\beta)\sigma_M(\varepsilon). \tag{49}
$$

The stresses σ_E and σ_M can be described by the relations (17) and (47), and we can therefore specialize the above expression for the strain ε belonging to each of the sets E_i, $i = 0, 1, 2, 3$:

$$
\begin{aligned}
&\sigma = \lambda \operatorname{tr} \varepsilon \underline{1} + 2\mu\varepsilon && \forall \varepsilon \in E_0, \\
&\sigma_q = \left[\beta\lambda + (1-\beta)\frac{2\lambda\mu}{\lambda+2\mu}\right](\varepsilon_i+\varepsilon_j) + \beta\lambda\varepsilon_k + 2\mu\varepsilon_q, \quad q = i,j && \\
&\sigma_k = \beta[\lambda(\varepsilon_1+\varepsilon_2+\varepsilon_3)+2\mu\varepsilon_k] && \forall \varepsilon \in E_1, \\
&\sigma_i = \left[\beta(\lambda+2\mu) + (1-\beta)\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\right]\varepsilon_i + \beta\lambda(\varepsilon_j+\varepsilon_k) && \\
&\sigma_q = \beta[\lambda(\varepsilon_1+\varepsilon_2+\varepsilon_3)+2\mu\varepsilon_q], \quad q = j,k && \forall \varepsilon \in E_2, \\
&\sigma = \beta(\lambda \operatorname{tr} \varepsilon \underline{1} + 2\mu\varepsilon) && \forall \varepsilon \in E_3.
\end{aligned} \tag{50}
$$

2.4. *Degradation of the material: the phase transformation as a dissipative process*

The state variable β describes the proportions of the two phases in the ceramic material. The degradation of the material under high tensile loading can be thought of as a phase transformation, from the E-phase to the M-phase, which is governed by the balance law (5). In our study, the dependence of the thermodynamical forces F and $\mathbf{H}$—appearing in (5)—upon the state variables β and $\nabla\beta$ is established by considering, for instance, that the free energy Ψ of the material is given by:

$$
\Psi(\varepsilon, \beta, \nabla\beta) = \beta\Psi_E(\varepsilon) + (1-\beta)\Psi_M(\varepsilon) + z(1-\beta) + I(\beta) + \frac{a}{2}\beta^2 + \frac{k}{2}\nabla\beta\cdot\nabla\beta, \tag{51}
$$

where a, z and k are positive material constants, highlighting the fact that we must have $a < z$. The indicator function $I(\beta)$ of the set [0, 1] is defined as:

$$I(\beta) = \begin{cases} 0, & \text{if } \beta \in [0, 1], \\ +\infty & \text{otherwise.} \end{cases} \tag{52}$$

As will be illustrated in the example in Section 3, the material constant z describes the stored energy density level that must be reached in a virgin material in order to start the phase transformation process. The indicator function $I(\beta)$ prevents the volume fraction β of the elastic phase from attaining values outside the interval $[0, 1]$. The term $(a/2)\beta^2$ is included in order to regularize the evolution of β along the process and finally the term $(k/2)\nabla\beta\cdot\nabla\beta$ is considered so as to give to β a diffusive behavior, thus smoothing the field β on Ω.

From (51), we define the following constitutive relations called the state laws:

$$\sigma = \frac{\partial\Psi}{\partial\varepsilon} = \beta\sigma_{\mathrm{E}}(\varepsilon) + (1-\beta)\sigma_{\mathrm{M}}(\varepsilon), \tag{53}$$

$$\mathbf{H} = \frac{\partial\Psi}{\partial\nabla\beta} = k\nabla\beta, \tag{54}$$

$$F^{\mathrm{R}} \in \partial_\beta\Psi = \Psi_{\mathrm{E}}(\varepsilon) - \Psi_{\mathrm{M}}(\varepsilon) - z + a\beta + \partial I(\beta), \tag{55}$$

where F^{R} denotes the reversible part of the thermodynamic force F defined in (2) and $\partial I(\beta)$ is the subdifferential of the indicator function I, at β.

To complete the constitutive equations, additional information about the dissipative nature of the phase transformation must be given. These informations can be obtained from the pseudo potential of dissipation Φ written as:

$$\Phi(\dot\beta) = i_{-}(\dot\beta), \tag{56}$$

where i_{-} is the indicator function of the set $(-\infty, 0]$. Since our model describes a simple material, the rate $\dot\beta$ describes, in a certain sense, the recent history of the function β. Thus, the time derivative considered here is a left derivative. In this study, we assume that $\dot\beta$ is a sectionally continuous function of the time, hence leading to a continuous history for the state variable β.

The irreversible part F^{Ir} of the force F is related to $\dot\beta$ by the constitutive relation, called the complementary law:

$$F^{\mathrm{Ir}} = F - F^{\mathrm{R}} \in \partial\Phi(\dot\beta) - \partial i_{-}(\dot\beta). \tag{57}$$

The irreversibility constraint $\dot\beta \leqslant 0$ is taken into account in the theory by the potential Φ and the complementary law (57), since it would be necessary to have infinite energy in order to have a negative $\dot\beta$.

3. ANALYSIS OF THE PHASE TRANSFORMATION MODEL

The state laws (53)–(55) and the complementary law (57) form a complete set of constitutive equations that model the behavior of ceramic-like materials.

The substitution of (54), (55) and (57) in (5) gives the following balance equation that governs the evolution of the volume fraction β of the elastic constituent:

$$\Psi_{\mathrm{E}}(\varepsilon) - \Psi_{\mathrm{M}}(\varepsilon) - z + a\beta + h + h_{-} + k\Delta\beta + l = 0, \tag{58}$$

where $h \in \partial I(\beta)$ and $h_{-} \in \partial i_{-}(\dot\beta)$.

Let us analyse the behavior of β as it obeys the balance law given by the expression (58). For the sake of simplicity, we shall consider the evolution of a bar of length L and constant cross-section submitted to imposed displacements at its ends. Furthermore, we shall suppose that the external thermodynamical force l is equal to zero.

Obviously, the field β will depend on the choice of the initial and boundary conditions. If we have $g(x = 0, t) = g(x = L, t) = 0$, then, from eqns (6) and (54), it results that:

$$\frac{\partial \beta}{\partial x}(x = 0, t) = \frac{\partial \beta}{\partial x}(x = L, t) = 0, \tag{59}$$

where x is the longitudinal coordinate. The following initial conditions associated with a virgin, unstrained material will be adopted:

$$\beta(t = 0) = 1, \quad \dot{\beta}(t = 0) = 0, \quad \varepsilon(t = 0) = 0. \tag{60}$$

The initial conditions above satisfy the balance equation (58). Indeed, under the above conditions, since $\Psi_E(0) = \Psi_M(0) = 0$ and the material constants are such that $-z+a$ is negative, then there are many pairs of positive elements h from $\partial I(1)$ and of positive elements h_- from $\partial i_-(0)$ such that:

$$-z+a+h+h_- = 0 \tag{61}$$

thus satisfying the balance equation (58).

Under the hypothesis of a uniaxial state of stress, the governing equation (58) can be reduced to:

$$\Psi_E(\varepsilon) - \Psi_M(\varepsilon) - z + a\beta + h + h_- + k\frac{\partial^2 \beta}{\partial x^2} = 0, \tag{62}$$

where, from (7) and (46):

$$\Psi_E = \frac{1}{2}\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\varepsilon_{xx}^2 = \tfrac{1}{2}E\varepsilon_{xx}^2 \tag{63}$$

and

$$\Psi_M = \begin{cases} 0, & \text{if } \varepsilon_{xx} > 0, \\ \tfrac{1}{2}E\varepsilon_{xx}^2, & \text{otherwise.} \end{cases} \tag{64}$$

From (62)–(64), we conclude that the phase parameter β cannot vary under compressive loads since then the difference $\Psi_E - \Psi_M$ is equal to zero, leading to the same situation expressed in (61) for the unstrained state.

If we consider the imposed displacements:

$$u_x(x = 0, t) = 0, \quad u_x(L, t) = \alpha t, \quad \alpha > 0, \quad \forall t \geq 0, \tag{65}$$

where u_x is the component of the displacement vector u in the axial direction, then eqn (62)—together with the initial and boundary conditions (60) and (61)—will admit only one homogeneous solution $\beta(x, t) = \beta_H(t)$ such that:

(i) We have $\beta_H = 1$ up to the instant:

$$t = t^* = \frac{L}{\alpha}\sqrt{\frac{2(z-a)}{E}}. \tag{66}$$

Indeed, the phase transformation parameter β_H can never increase its value since otherwise the rate $\dot{\beta}_H$ would be positive, the set $\partial i_-(\dot{\beta}_H)$ would be void and the expression (62) would become meaningless. This behavior characterizes the irreversible nature of the present phase transformation. On the other hand, the parameter β_H cannot decrease its value (leading to a phase transformation) whenever we have:

$$\Psi_E(\varepsilon) - z + a \leqslant 0 \tag{67}$$

because if we have decided that $\dot{\beta}_H$ is negative, there would be a left neighborhood V_- at the instant t_0 where for any $t_0+\Delta t$, $\Delta t < 0$, belonging to V_- the parameter β would have assumed values greater than one, which could not have happened (or otherwise the set $\partial I(\beta)$ would be void). Thus, whenever we have (67), the rate $\dot{\beta}_H$ must be null. This means that the material does not degrade in such a situation.

(ii) After the time instant $t = t^*$, the term $\Psi_E(e) - z + a$ becomes positive and the balance equation becomes:

$$\Psi_E(\varepsilon) - z + a\beta_H + h + h_- = 0, \tag{68}$$

which can be satisfied only if β_H decreases so as to satisfy:

$$\Psi_E(\varepsilon) - z + a\beta_H = 0, \quad h = 0 \quad \text{and} \quad h_- = 0. \tag{69}$$

This is verified for:

$$\beta_H = \frac{z - \Psi_E(\varepsilon)}{a}, \quad 0 < \beta_H < 1, \quad \dot{\beta}_H \leqslant 0. \tag{70}$$

If we consider the constitutive law given by (63) and the load history given by (65), we can rewrite the evolution (70) of β_H as:

$$\beta_H = \frac{1}{a}\left[z - \frac{1}{2}E\left(\frac{\alpha t}{L}\right)^2\right], \quad t^* \leqslant t \leqslant t^{**} = \frac{L}{\alpha}\sqrt{\frac{2z}{E}}. \tag{71}$$

The phase parameter reaches zero at time t^{**} defined above, characterizing a totally broken material.

(iii) As the imposed displacement continues to increase, the term $\Psi_E(\varepsilon) - z$ becomes positive, and the balance equation reduces to:

$$\Psi_E(\varepsilon) - z + h + h_- = 0, \tag{72}$$

which is clearly satisfied due to the existence of a negative element $h \in \partial I(0)$ and a non-negative element $h_- \in \partial i_-(\beta_H)$.

Figure 3 shows the uniaxial stress–strain curve for this tensile test. The material constants z and a can be obtained experimentally by comparing the uniaxial strains ε^*_{xx} and ε^{**}_{xx} with the analytical values

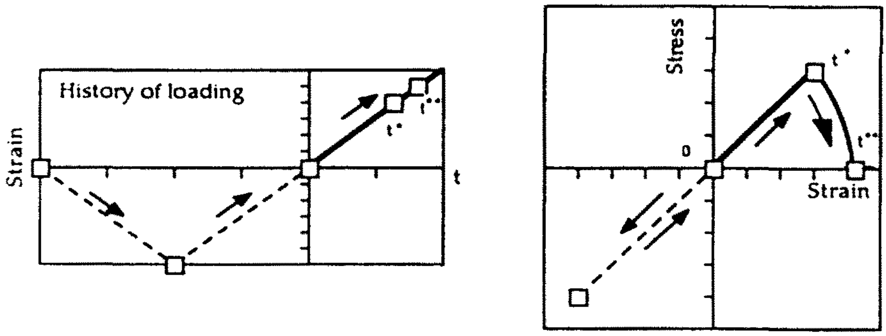

Fig. 3. Stress–strain curve for the tensile test.

$$\varepsilon_{xx}^{*} = \sqrt{2\frac{(z-a)}{E}}, \quad \varepsilon_{xx}^{**} = \sqrt{\frac{2z}{E}}. \tag{73}$$

If we consider the following imposed displacement:

$$u_x(L,t) = \begin{cases} \alpha t & \text{until } t = t_1 \text{ with } t^* < t_1 < t^{**}, \\ \alpha(2t_1 - t) & \text{for } t \geqslant t_1, \end{cases} \tag{74}$$

we have the uniaxial stress–strain curve shown in Fig. 1.

4. A THERMODYNAMICALLY CONSISTENT MODEL

A consistent constitutive theory must not admit processes where the dissipation rate d [see eqn (12)] is negative.

Proposition 4.1. *The constitutive model defined by the potentials* $\Psi(\varepsilon,\beta,\nabla\beta)$ *in* (51) *and* $\Phi(\beta)$ *in* (56) *is consistent with the Second Law of Thermodynamics expressed by* (13).

Proof. The rate of variation of the potential Ψ is given by:

$$\Psi(\varepsilon,\beta,\nabla\beta) = \frac{\partial\Psi}{\partial\varepsilon}\cdot\dot{\varepsilon} + \frac{\partial\Psi}{\partial\nabla\beta}\cdot\nabla\beta + (\Psi_E(\varepsilon) - \Psi_M(\varepsilon) - z + \alpha\beta)\beta$$
$$+ \lim_{\Delta t\to 0}\left[\frac{I(\beta(t+\Delta t)) - I(\beta(t))}{\Delta t}\right], \tag{75}$$

where Δt is negative since the rate Ψ is a left derivative; the limit is to be computed at the instant t with the information which is available.

The subdifferential $\partial I(\beta(t))$ is such that:

$$I(\beta(t+\Delta t)) - I(\beta(t)) \geqslant h(\beta(t+\Delta t) - \beta(t)) \quad \forall h \in \partial I(\beta(t)). \tag{76}$$

The division of (76) by $\Delta t < 0$ gives:

$$\frac{I(\beta(t+\Delta t)) - I(\beta(t))}{\Delta t} \leqslant h\frac{\beta(t+\Delta t) - \beta(t)}{\Delta t} \quad \forall h \in \partial I(\beta(t)). \tag{77}$$

The limit of the inequality (77) as $\Delta t < 0$ approaches the value zero leads to:

$$\lim_{\Delta t\to 0}\frac{I(\beta(t+\Delta t)) - I(\beta(t))}{\Delta t} \leqslant h\beta \tag{78}$$

and then we have:

$$\Psi(\varepsilon,\beta,\nabla\beta) \leqslant \sigma\cdot\dot{\varepsilon} + \mathbf{H}\cdot\nabla\beta + F^{\mathrm{R}}\beta \quad \forall h \in \partial I(\beta(t)), \tag{79}$$

where the constitutive relations (53)–(55) were taken into account. The intrinsic dissipation d then verifies:

$$d = \sigma\cdot\dot{\varepsilon} + (F^{\mathrm{R}} + F^{\mathrm{Ir}})\beta + \mathbf{H}\cdot\nabla\beta - \Psi \geqslant F^{\mathrm{Ir}}\beta. \tag{80}$$

On the other hand, from the definition of $\partial i_{-}(\beta)$ we have:

$$\partial i_-(\beta) = \{F^{lr} \in R : i_-(\beta^*) - i_-(\beta) \geqslant F^{lr}(\beta^* - \beta) \quad \forall \beta^* \in R\}. \tag{81}$$

In particular, for $\beta^* = 0$, every element F^{lr} from $\partial i_-(\beta)$ is such that:

$$F^{lr}\beta \geqslant i_-(\beta) = 0 \quad \forall \beta \leqslant 0. \tag{82}$$

Hence, from (f) and (h) we conclude that the dissipation is always non-negative and the present model is consistent with the Second Law of Thermodynamics.

5. CONCLUSIONS

We proposed a mechanical model which presents the main features of a ceramic material, in the sense that it shows a brittle behavior under tensile loading, while verifying an unlimited strength under compressive efforts. The material was supposed to be a mixture of an elastic phase and a masonry phase, the rupture being associated with a phase change from the former to the latter one. The model was shown to be consistent with the Thermodynamics of Irreversible Processes.

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APPENDIX: ISOTROPY OF MASONRY MATERIALS

The isotropy property of the constitutive law (14) is useful for computing the energy function $\psi_M(\varepsilon)$: as in such cases the expression (16) can be rewritten in terms of the eigenvalues of ε and σ, the imposition of the constraints on the stresses becomes easier to perform.

Let $\mathcal{Y}$ express the collection of all subsets of R. Let us denote the set of all linear operators on a vector space into itself by Lin. Let Ort $\subset$ Lin be the set of all orthogonal linear operators and let $\mathcal{X}$ be the collection of all subsets of Lin.

First, we require an extension of the classical definition of isotropy to the case where the function is set valued.

Definition A.1. A multivalued function F: Lin $\rightarrow Y$ is *isotropic* if

$$F(\mathbf{QAQ}^T) = F(\mathbf{A}) \quad \forall \mathbf{A} \in \mathrm{Lin}, \quad \forall \mathbf{Q} \in \mathrm{Ort}. \tag{A1}$$

A multivalued function $\mathbf{G}$: Lin $\rightarrow \mathcal{X}$ is said to be *isotropic* if

$$\mathbf{QG}(\mathbf{A})\mathbf{Q}^T = \mathbf{G}(\mathbf{QAQ}^T) \quad \forall \mathbf{A} \in \mathrm{Lin}, \quad \forall \mathbf{Q} \in \mathrm{Ort}, \tag{A2}$$

where

$$\mathbf{QG}(\mathbf{A})\mathbf{Q}^T = \{\mathbf{f} = \mathbf{QgQ}^T ; \mathbf{g} \in \mathbf{G}(\mathbf{A})\}. \tag{A3}$$

Proposition A.1. *The indicator function $I_-(\boldsymbol{\sigma}_M)$ is isotropic.*

Proof. The function under consideration is isotropic if

$$I_-(\mathbf{Q}\boldsymbol{\sigma}_M\mathbf{Q}^T) = I_-(\boldsymbol{\sigma}_M) \quad \forall \mathbf{Q} \in \mathrm{Ort} \tag{A4}$$

but (A4) is trivial since the effective domain of $I_-(\boldsymbol{\sigma}_M)$ is determined by the condition $\sigma_{M_i} \leqslant 0$, $i = 1, 2, 3$ and both $\boldsymbol{\sigma}_M$ and $\mathbf{Q}\boldsymbol{\sigma}_M\mathbf{Q}^T$ have the same eigenvalues.

Proposition A.2. *The function $\partial I_-(\boldsymbol{\sigma}_M)$ is isotropic.*

Proof. From the definition (A1), the function $\partial I_-(\boldsymbol{\sigma})$ is isotropic if

$$\partial I_-(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T) = \mathbf{Q}\,\partial I_-(\boldsymbol{\sigma})\mathbf{Q}^T \quad \forall \boldsymbol{\sigma} \in \mathrm{Sym}' \quad \forall \mathbf{Q} \in \mathrm{Ort} \tag{A5}$$

but

$$\begin{aligned} \partial I_-(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T) &= \{\mathbf{f} \in \mathrm{Sym}' ; \mathbf{f} \cdot (\dot{\boldsymbol{\tau}} - \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T) \leqslant I_-(\dot{\boldsymbol{\tau}}) - I_-(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T)\, \forall \dot{\boldsymbol{\tau}} \in \mathrm{Sym}'\} \\ &= \{\mathbf{f} \in \mathrm{Sym}' ; \mathbf{f} \cdot \mathbf{Q}(\boldsymbol{\tau} - \boldsymbol{\sigma})\mathbf{Q}^T \leqslant I_-(\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T) - I_-(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T)\, \forall \boldsymbol{\tau} \in \mathrm{Sym}'\}, \end{aligned} \tag{A6}$$

where $\boldsymbol{\tau} = \mathbf{Q}\dot{\boldsymbol{\tau}}\mathbf{Q}^T$. As $I_-(\boldsymbol{\sigma})$ is an isotropic function, we have $I_-(\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T) = I_-(\boldsymbol{\tau})$ $\forall \boldsymbol{\tau} \in \mathrm{Sym}'$, $\forall \mathbf{Q} \in \mathrm{Ort}$. Thus, it follows that:

$$\partial I_-(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T) = \{\mathbf{f} \in \mathrm{Sym}' ; \mathbf{Q}^T\mathbf{f}\mathbf{Q} \cdot (\boldsymbol{\tau} - \boldsymbol{\sigma}) \leqslant I_-(\boldsymbol{\tau}) - I_-(\boldsymbol{\sigma})\, \forall \boldsymbol{\tau} \in \mathrm{Sym}'\}. \tag{A7}$$

Let us define $\mathbf{g} = \mathbf{Q}^T\mathbf{f}\mathbf{Q}$. We can now rewrite the above expression as:

$$\begin{aligned} \partial I_-(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T) &= \{\mathbf{QgQ}^T \in \mathrm{Sym}' ; \mathbf{g} \cdot (\boldsymbol{\tau} - \boldsymbol{\sigma}) \leqslant I_-(\boldsymbol{\tau}) - I_-(\boldsymbol{\sigma})\, \forall \boldsymbol{\tau} \in \mathrm{Sym}'\} \\ &= \{\mathbf{QgQ}^T \in \mathrm{Sym}' ; \mathbf{g} \in \partial I_-(\boldsymbol{\sigma})\} \\ &= \mathbf{Q}\,\partial I_-(\boldsymbol{\sigma})\mathbf{Q}^T \end{aligned} \tag{A8}$$

and hence the assertion is true.

Proposition A.3. *The strain function $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ given by* (14) *is isotropic.*

Proof. By considering the result (A3), the expression (14) can be developed as follows:

$$\begin{aligned} \boldsymbol{\varepsilon}(\mathbf{Q}\boldsymbol{\sigma}_M\mathbf{Q}^T) &= \lambda^c(\mathrm{tr}\,\mathbf{Q}\boldsymbol{\sigma}_M\mathbf{Q}^T)\mathbf{1} + 2\mu^c\mathbf{Q}\boldsymbol{\sigma}_M\mathbf{Q}^T + \partial I_-(\mathbf{Q}\boldsymbol{\sigma}_M\mathbf{Q}^T) \\ &= \lambda^c(\mathrm{tr}\,\boldsymbol{\sigma}_M)\mathbf{QQ}^T + 2\mu^c\mathbf{Q}\boldsymbol{\sigma}_M\mathbf{Q}^T + \mathbf{Q}\,\partial I_-(\boldsymbol{\sigma}_M)\mathbf{Q}^T \\ &= \mathbf{Q}\boldsymbol{\varepsilon}(\boldsymbol{\sigma}_M)\mathbf{Q}^T. \end{aligned} \tag{A9}$$

Now we extend the Transfer Theorem [cf. Gurtin (1981)] to the case of multivalued functions.

Proposition A.4. *Let* $\mathbf{G} : \Gamma \rightarrow \mathcal{X}$, $\Gamma \subset$ Sym, *be isotropic. Then every eigenvector of* $\mathbf{A} \in \Gamma$ *is an eigenvector of* $\mathbf{G}(\mathbf{A})$.

Proof. Let $\mathbf{e}$ be an eigenvector of $\mathbf{A} \in \Gamma$, and let $\mathbf{Q} \in \mathrm{Ort}$ be the reflection across the plane perpendicular to $\mathbf{e}$:

$$\mathbf{Qe} = -\mathbf{e}, \quad \mathbf{Qf} = \mathbf{f} \quad \text{if} \quad \mathbf{f} \cdot \mathbf{e} = 0. \tag{A10}$$

Then by the spectral theorem, $\mathbf{Q}$ leaves invariant the characteristic spaces of $\mathbf{A}$. Hence we may conclude from the commutation theorem that:

$$\mathbf{QAQ}^T = \mathbf{A}. \tag{A11}$$

Thus, since $\mathbf{G}$ is isotropic,

$$\mathbf{QG}(\mathbf{A})\mathbf{Q}^T = \mathbf{G}(\mathbf{QAQ}^T) = \mathbf{G}(\mathbf{A}) \tag{A12}$$

or

$$\mathbf{Q}\mathbf{G}(\mathbf{A}) = \mathbf{G}(\mathbf{A})\mathbf{Q} \tag{A13}$$

which means that:

$$\mathbf{Q}\mathbf{g} = \mathbf{g}\mathbf{Q} \quad \forall \mathbf{g} \in \mathbf{G}(\mathbf{A}). \tag{A14}$$

Therefore,

$$\mathbf{Q}\mathbf{g}\mathbf{e} = \mathbf{g}\mathbf{Q}\mathbf{e} = -\mathbf{g}\mathbf{e} \quad \forall \mathbf{g} \in \mathbf{G}(\mathbf{A}). \tag{A15}$$

Since $\mathbf{ge}$ is transformed by $\mathbf{Q}$ exactly in the same manner as $\mathbf{e}$, we conclude that these vectors are parallel to each other for any $\mathbf{g}$ belonging to $\mathbf{G}(\mathbf{A})$:

$$\mathbf{g}\mathbf{e} = \alpha\mathbf{e}, \quad \alpha \in R, \quad \forall \mathbf{g} \in \mathbf{G}(\mathbf{A}) \tag{A16}$$

and hence $\mathbf{e}$ is an eigenvector of any $\mathbf{g} \in \mathbf{G}(\mathbf{A})$.